On Microfoundations of the City*

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Abstract

This paper considers the spatial structure of a city subject to final demand and vertical linkages. Individuals consume differentiated goods (or services) and firms purchase differentiated inputs (or services) in product (or service) markets where firms compete under monopolistic competition. Workers rent their residential lots in an urban land market and contribute to the production of differentiated goods and inputs. We show that firms and workers co-agglomerate and endogenously form a city. We characterize and discuss the spatial distribution of firms and consumers in such cities on one- and two-dimensional spaces. We show that final demand and vertical linkages raise the urban density and reduce the city spread.

Keywords: co-agglomeration, continuous distribution, new economic geography, forward and backward linkages, housing space.

J.E.L. Classification: C62, F12, R12.

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1 Introduction

Many international economists and economic geographers highlight the importance of market externalities in the location of economic activities between and within countries. They explain how the endogenous agglomeration of economic activities in specific locales can be the result of specific linkages. While forward linkages (or supply linkages) entice consumers to locate closer to producers in order to benefit from a larger diversity of less expensive products, backward linkages (or demand linkages) entice the producers to locate closer to their final consumers or client firms in order to save on transport costs (Krugman 1991, Krugman and Venables 1995). Such a literature contrasts with the urban economic literature that traditionally discusses the endogenous formation of cities in the light of non-market externalities. In particular, cities are seen as business information centers or social interaction centers that build on the agglomeration forces stemming from human interactions, face-to-face communications or exogenous technological spillovers (Beckmann, 1976). In this context, agents have incentives to locate close to each other because proximity to other agents increases the efficiency of communications and interactions.

Non-market face-to-face interaction is certainly an important factor of the formation of many central business districts, but it is probably not the sole driver of the economic agglomeration process within most city centers. For instance, Fujita and Tabuchi (1997) study how face-to-face interactions take place within headquarters and central functions of Japanese firms and how those interactions entice firms to locate those functions in the centers of the largest Japanese cities, mainly Tokyo and Osaka. Nevertheless, many other activities in the firms’ branches are shown to take place in other cities or at less central locations of Tokyo and Osaka. Face-to-face interactions between headquarters can therefore not be the unique agglomeration driver in Japanese cites.

Rather, we highlight forward and backward linkages as important drivers of city structures. First, there indeed is ample evidence of vertical linkages (or input-output linkages) in cities as most cities consume a significant share of their own production. For example, in 2000, a prefecture like Tokyo sold some 33% and 30% of its production to its own residents and firms whereas it sold only 7% and 29% to residents and firms outside the prefecture. Similar figures can be obtained for many metropolitan areas. Second, large cities generate a significant share of the product and service demand for their own firms. To emphasize this point, Mathae and Shwachman (2009) compute the market potentials of
each European region (i.e. transportation-cost-weighted market capacities at NUT2 level) and break down those according to the share generated by the region itself, its neighboring regions, and the rest of EU regions. Those authors show that a region that hosts a large city strongly contributes to the market potential for its own firms. For example, the Inner London contributes up to 78% of the market potential that its own firms have access to.\(^1\) Finally, trade microeconometric studies confirm the idea that market transactions mainly occur within the same urban locations. As a point in case, Hilberry and Hummels (2008) show the large bulk of trade between firms occurs within a median radius of just 4 miles.

This paper investigates the spatial structure of a city subject to backward and vertical linkages. We consider endogenous urban location of individuals and firms who consume the set of differentiated goods or services that they produce. We present a model where residents have hyperbolic preferences for residential space and quadratic preferences for differentiated varieties of goods or services and where residents work in the firms that produce and sell those varieties under monopolistic competition. Firms hire their workforce around their production sites while they ship and sell products or offer their services directly to consumers. This set-up allows us to study how the urban structure is shaped by forward and backward linkages: firms move to urban locations where intermediate goods or production factors are supplied with low prices and firms are attracted by locations where their intermediate or final goods are highly demanded.

Due to the linkages, firms prefer to locate and hire labor in locales closer to the consumers and the other firms that also purchase their intermediate output but that eventually compete with them. As a result, firms and workers tend to co-agglomerate and form a city center endogenously. We characterize and discuss the shape of the residential distribution in such cities on one- and two-dimensional Euclidean spaces (linear and planar cities). We show that residential distribution is continuous, symmetric and single-peaked. To our knowledge, this is the first formal model of urban spatial structure based on a microeconomic foundation of forward and backward linkages as it has been discussed in the framework of new economic geography. We furthermore disentangle the impact of supply linkages from final demand linkages and show that both effects foster the concentration of firms and raise urban land rents.

\(^1\) Similarly, the share of own market potential is larger than 70% in the regions (NUTS2) hosting Berlin, Brussels, Hamburg, Madrid, Praha, Paris, Stockholm and Helsinki. We thank Th. Mathae for providing this information.
The paper relates to the literature as follows. Since the pioneering work of von Thünen (1826), the study of urban structure has often focused on the assumption of central business districts, where firms locate on a spaceless point as in Alonso’s (1964) residential location theory and on a set of such points as in Fujita, Krugman and Mori’s (1995) new economic geography. Although this literature has discussed the impact of linkages on the spatial distribution of cities, the spatial distribution of firms and workers inside a city remains an unsolved issue. On the other hand, the formation of city and the distribution of urban activities within a city have been the focus of a small set of contributions initiated by Beckmann (1976) and followed by O’Hara, (1977), Ogawa and Fujita (1980), Fujita and Ogawa (1982), Tauchen and Witte (1984), Fujita (1988), Tabuchi (1986), Lucas and Rossi-Hansberg (2001), Berliant et al. (2002), Berliant and Wang (2008), and Mossay and Picard (2011). These studies have paid attention to the impact of social interactions and technological externalities on urban structure mainly through face-to-face communications that enhance the productivity of pairs of firms or workers. Whereas the production mechanisms of face-to-face communications modeled in those studies have produced convenient analytical properties, they remain black boxes because they do not take into account the actual nature of economic interactions within a city. In a similar vein, Lucas and Rossi-Hansberg (2002) model the positive pecuniary externalities between the firms as exogenous spillovers that decay with distance. To our knowledge, none of those contributions have opened the black box of the externalities accruing between the firms that choose to locate in a same city. By contrast, this paper presents some microeconomic foundation of the pecuniary externalities that generate backward and forward linkages developed by Krugman (1991), Ottaviano et al. (2002) and Krugman and Venables (1995). In such microeconomic foundations, the prices of goods and services are determined in general equilibrium under monopolistically competitive markets.

The new economic geography literature offers only a small set of theoretical studies on firms’ spatial distribution over a continuous space. The scarcity of such a research mainly results from the analytical difficulties involved in the study of location incentives on a spatial continuum. In particular, Fujita, Krugman and Venables (1999) and Mossay (2003) have reduced the study to uniform distributions of firms (flat earth) while Picard and Tabuchi (2010) have enlarged it to a larger class of spatial distributions. Because of the absence of preference for residential space, those authors find that continuous equilibrium distributions are most often intrinsically unstable. This suggests that economic activities must agglomerate in spaceless points, which are then called cities. By contrast, this paper
introduces preferences for residential space and shows existence of a unique equilibrium distribution of firms and workers, that is continuous, symmetric and single-peaked.

The remainder of the paper is organized as follows. Section 2 presents the urban model with final demand and vertical linkages. Section 3 presents the short run equilibrium while Section 4 characterizes long run spatial equilibrium. Sections 5 and 6 apply this discussion to the one- and two-dimension Euclidean space set-ups, respectively. Section 7 concludes. Proofs are relegated to the Appendix.

2 The Model

We assume two sets of perfectly mobile individuals and differentiated product varieties whose residence and production are distributed on a geographical compact space $\mathcal{X} \subseteq \mathbb{R}^X$, $X = 1, 2$. On the one hand, the mass of individuals residing at location $x$ is defined by the density function $\lambda(x)$, where $x \in \mathcal{X}$ is the individual’s coordinate. Without loss of generality, we assume that the total mass of individuals is unity: $\int_{\mathcal{X}} \lambda(x)dx = 1$. On the other hand, each variety is produced at a single location with coordinate $y \in \mathcal{X}$ while the mass of varieties produced at location $y$ is defined by the distribution function $\mu(y)$. The total mass of varieties is given by $M = \int_{\mathcal{X}} \mu(y)dy$. Product varieties can be interpreted as services. In the case of a two-dimensional Euclidean space $\mathcal{X}$, $x$ and $y$ are the vectors of coordinates $(x_1, x_2)$ and $(y_1, y_2)$ whereas the notations $\lambda(x)dx$ and $\mu(y)dy$ are equivalent to $\lambda(x_1, x_2)dx_1dx_2$ and $\mu(y_1, y_2)dy_1dy_2$.

2.1 Consumers’ preferences and demands

Individuals consume the differentiated product varieties, residential space and a numéraire good. The preferences of an individual located at location $x$ are given by the following utility function:

$$U[c_0(x), c_m(\cdot, x), s(x)] = C[c_m(\cdot, x)] - \frac{\theta}{2} \frac{1}{s(x)} + c_0(x)$$

(1)

where $c_0(x)$ is the consumption for a homogenous good used as numéraire, $s(x)$ is the consumption for space, $c_m(\cdot, x)$ is the consumption profile for differentiated product varieties that are offered at location $x$, and $C[c_m(\cdot, x)]$ is a composite good function aggregating those product varieties. The parameter $\theta$ reflects the preference for residential space, a larger $\theta$ implying a stronger preference for space. The preference for space reflects a decreasing marginal utility from use of residential space.\footnote{The present hyperbolic preference for space and the Beckmann’s (1976) logarithmic preference for space are two instances of the same class of preferences $(s^{1-\rho} - 1)/(1 - \rho)$ where $\rho = 2$ and $\rho \to 1$ respectively, which yield iso-elastic}
The preference for residential space acts as a dispersion force in this model.\footnote{In new economic geography, the dispersion force is usually obtained by the assumption of productive land that creates a localized demand for workers in the farming sector. This analytically convenient assumption is often criticized on the ground that farming products must be undifferentiated and transported at zero cost (see e.g. Fujita, Krugman and Venables 1999, Picard and Zeng 2005).}

As in Ottaviano et al. (2002) the composite good is expressed as

$$C\left[c_m(\cdot, x)\right] = \alpha \int_X c_m(y, x)\mu(y)dy - \frac{\beta}{2m} \int_X [c_m(y, x)]^2 \mu(y)dy - \frac{\gamma}{2m} \left[ \int_X c_m(y, x)\mu(y)dy \right]^2$$

This composite good is made of individual consumption $c_m(y, x)$ of a variety produced at location $y$ and consumed at location $x$. In the above expression, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are parameters reflecting the preference for the goods or services. Ceteris paribus, a higher $\alpha$ implies a more intense preference towards the varieties compared to the numéraire, a higher $\beta$ means a more bias toward a dispersed consumption of varieties (i.e. the love for variety), and a higher $\gamma$ implies a higher degree of substitutability between varieties. Finally, we introduce the consumer demand multiplier $m$. It will be shown in Section 3.1 that the consumer demand and surplus are proportional to this multiplier $m$.

The budget constraint of an individual located at $x$ is equal to

$$\int_X p(y, x)c_m(y, x)\mu(y)dy + R(x)s(x) + c_0(x) \leq w(x) + \tau_0 \tag{2}$$

where $p(y, x)$ is the price of a variety produced at location $y$ sold at location $x$, $R(x)$ is the land rent, $w(x)$ is the individual’s income when she resides at location $x$, and $\tau_0$ is her initial endowment of numéraire. We assume that the endowment $\tau_0$ is large enough so that consumers have positive demands for the numéraire in equilibrium. We also assume that product varieties are exchanged for any configurations of firms and consumers. We now present the production side of the economy.

### 2.2 Production

Each firm produces a single variety of goods or services, sets the prices of its goods or services and delivers them to the consumer locations incurring a transport cost. Each firm faces a monopolistic demands for residential space with price elasticity respectively equal to $1/2$ and $1$. So, the present hyperbolic preference represents an intermediate setting between Beckmann’s demand and the inelastic demand for residential space that is regularly used in standard urban economics (e.g. Fujita and Ogawa 1980). As will be shown below, the hyperbolic sub-utility for space yields more convenient analytical properties.
competition in the following sense. First, it faces price competition with other firms that sell imperfect
substitutes. Second, because the mass of each firm is negligible in the whole market, it cannot determine
its price strategically. Finally, there exist many potential firms that enter until profits fall to zero.

As in most modern cities, each firm uses some intermediate inputs that are produced in the city. Like Krugman and Venables (1995), we make things simple by assuming that firms use the same set
of differentiated goods or services as those purchased by final consumers and that their benefits from
using those goods or services take the same form as the consumers’ preferences. To be more precise,
we now assume that production requires no variable inputs (without loss of generality as in Ottaviano
et al. 2002) but that it requires three different fixed inputs: labor, physical capital equipment and
intermediate goods or services. We assume that, a firm producing at location \( y \) must hire an inelastic
unit of labor and pay a wage \( w(y) \). We assume that a too high commuting cost to refrain workers
to commute. Hence, each worker resides at the same place as its firm.\(^4\) As a result, when the labor
market clears, the mass of varieties is equal to the mass of workers: \( M = 1 \) and \( \mu(y) = \lambda(y) \).

In addition, to build up and use its production equipment, the firm must acquire some physical
capital which costs \( K \) units of numéraire. Alternatively, the firm can buy \( c_k(\cdot, y) \) units of intermediate
goods at a price \( p(\cdot, y) \) to reduce its cost of physical capital or operation. Physical capital and inter-
mediate goods are therefore input substitutes. One interpretation is that a part of the physical capital
can be replicated by a set of intermediate inputs at a lower cost. More specifically, the use of a set
of \( c_k(\cdot, y) \) intermediate inputs reduces the requirement for physical capital to \( K - C[c_k(\cdot, y)] \) units of
numéraire where \( C[c_k(\cdot, y)] \) is assumed to take the same form as the composite good in the consumers’
preferences. In this case, the demand multiplier \( m \) for the consumer demand is simply replaced by
the input-output multiplier \( k \) that applies on the demand for intermediate input. Such an assumption
replicates Krugman and Venables’ (1995) assumption on firms’ production functions to Ottaviano et
al.’s (2002) quadratic utility model. Note that firms use no space.

Given this set-up, the firm located at \( y \) maximizes a profit \( \Pi(y) = e(y) - f(y) \) that embeds the
operating profit

\[
e(y) \equiv \int_X [p(y, x) - \tau(y, x)] [c_m(y, x)\lambda(x) + c_k(y, x)\mu(x)] \, dx
\]

\(^4\)Picard and Tabuchi (2010) discuss commuting patterns in the same model. See also Ogawa and Fujita (1980) and
Fujita and Ogawa (1982), among others, for spatial structure with commuting.
and the fixed cost
\[ f(y) \equiv K - C[c_k(\cdot, y)] + \int_{\mathcal{X}} p(z, y) c_k(z, y) \lambda(z) dz + w(y) \]  
(4)

where \( c_m(y, x) \) is the demand of a good produced at location \( y \) by consumers at location \( x \) and \( \tau(y, x) \) is the unit transport cost from locations \( y \) to \( x \). Note that each firm delivers a good rather than buyers visit firms, and that each firm exercises discriminatory prices rather than charges the same mill price. The firm thus makes two choices: one about its prices \( p(y, \cdot) \) and one about its own demand \( c_k(\cdot, y) \) of intermediate inputs to other firms. Because the former decision affects operating profits and the latter fixed costs, the two decisions can be disentangled into the maximization of operating profits and the minimization of fixed costs.

We first discuss the short-run equilibrium where land, product and labor markets clear. We then discuss the long run spatial equilibrium where firms and workers relocate within the city.

3 Short run equilibrium

In this section we consider the individuals’ demands for products or services and for residential space, the firms’ demands for intermediate inputs and finally the market prices and profits. We finally determine the individual’s location incentives within the city.

3.1 Product and land demands

Following the tradition of urban economic models à la Alonso (1964), assume that land is owned by absentee landlords who do not participate in production and consumption. As usual in the urban economic literature, the land market equilibrium can be discussed by describing the behavior of landlords who dictate the residents’ consumption behaviors. At each location \( x \), landlords extract the maximum land rent that does not entice residents to move away from their parcels. In equilibrium residents have no incentives to relocate and reach the same utility level. So, at the equilibrium utility level \( U^* \), landlords at \( x \) collect a bid rent

\[ \Psi(x) = \max_{s(x), c_m(\cdot, x), c_p(x)} \frac{w(x) + \bar{c}_0(x) - \int_{\mathcal{X}} p(y, x) c_m(y, x) \mu(y) dy}{s(x)} \text{ s.t. } U[c_0(x), c_m(\cdot, x), s(x)] \geq U^* \]

which is the largest per-square-meter expenditure on residential space that does not entice residents to move away. Since utility increases in the numéraire \( c_0(x) \), the above constraint binds so that we
can write the above bid rent function as
\[ \Psi(x) = \max_{s(x), c_m(\cdot, x)} \frac{w(x) + \bar{c}_0 - \frac{\theta}{2} \frac{1}{s(x)} - U^* + C [c_m (\cdot, x)] - \int_{\mathcal{X}} p(y, x) c_m(y, x) \mu(y) dy}{s(x)} \]

The final demands for varieties of goods and services \( c_m (\cdot, x) \) are the solutions of this maximization problem. We get \( c_m(y, x) = mc^*(y, x) \) where
\[ c^*(y, x) = \frac{\alpha}{\beta + \gamma} - \frac{1}{\beta} p(y, x) + \frac{\gamma}{\beta (\beta + \gamma)} P(x) \]
\[ P(x) = \int_{\mathcal{X}} p(y, x) \mu(y) dy \]
and is the price index for consumers at location \( x \) (see Picard and Tabuchi, 2010). Similarly, the consumer surplus that an individual located at \( x \) obtains from consuming the differentiated goods is given by \( mS^* [p (\cdot, x)] \) where
\[ S^* [p (\cdot, x)] = \frac{\alpha^2}{2 (\beta + \gamma)} - \frac{\alpha}{\beta + \gamma} \int_{\mathcal{X}} p(y, x) \mu(y) dy \]
\[ - \frac{\gamma}{2 \beta (\beta + \gamma)} \left[ \int_{\mathcal{X}} p(y, x) \mu(y) dy \right]^2 + \frac{1}{2 \beta} \int_{\mathcal{X}} [p(y, x)]^2 \mu(y) dy \]
which depends on the price profile \( p (\cdot, x) \). Thus, the consumer demand and surplus are simply proportional to the demand multiplier \( m \). Plugging those equilibrium demand and surplus values in the utility function yields
\[ V(x) = w(x) + mS^* [p (\cdot, x)] - \theta \frac{1}{s(x)} + \bar{c}_0 \] (5)
and the bid rent function simplifies to
\[ \Psi(x) = \max_{s(x)} \frac{mS^* [p (\cdot, x)] + w(x) + \bar{c}_0 - \frac{\theta}{2} \frac{1}{s(x)} - U^*}{s(x)} \] (6)
The demand for residential space is then the solution of this problem: \( s(x) = \theta/[mS^* [p (\cdot, x)] + w(x) + \bar{c}_0 - U^*] \). Finally, the equilibrium rent is equal to the bid rent: \( R(x) = \Psi(x) \). Hence, the land rent is determined as
\[ R(x) = \frac{\theta}{2} \frac{1}{s(x)^2} \] (7)

We now turn to the firms’ demand for intermediate inputs and to their prices decisions.

3.2 Intermediate input demands and equilibrium prices

The firm’s cost minimization has the same form as the consumer’s utility maximization. It therefore yields a firm’s demand of \( c_k(y, x) = kc^*(y, x) \) units of intermediate inputs, where \( k \) is the input-output multiplier. Similarly, cost minimization yields a capital saving equal to \( kS^* [p (\cdot, y)] \) so that the
minimized fixed cost (4) is then given by

\[ K - kS^* [ \rho (\cdot , y) ] + w(y) \]  

(8)

That is, the minimized fixed cost is equal to the cost of physical capital \( K \) minus the cost savings \( kS [ \rho (\cdot , y) ] \) plus the wages to workers \( w(y) \). The demand for intermediate inputs and their corresponding cost savings are proportional to the consumer’s demand and surplus.

We can now establish the prices that firms set for their consumers and client firms. The demand addressed to each firm located at \( y \) is equal to \( q(y, x) \equiv (m + k) c^*(y, x) \). The firm located at \( y \) finds the price profile \( p(y, \cdot ) \) that maximizes it operating profit \( e(y) \) in (3). The first-order condition for profit maximization yields the optimal price of variety produced at location \( y \) and sold to a consumer residing at location \( x \):

\[ p^*(y, x) = \bar{p}(x) + \frac{1}{2} \tau (y, x) \quad \text{where} \quad \bar{p}(x) = \frac{2\alpha\beta + \gamma \int_X \tau (y, x) \lambda(y) dy}{2(2\beta + \gamma)} \]  

(9)

Since the parameters \( m \) and \( k \) have a multiplicative effect on demand, they have the same multiplicative effect on the operating profits. As a result, the equilibrium prices (9) are invariant to the demand and input-output multipliers. At the price equilibrium the demand is equal to \( q(y, x) = (m + k) [ p^*(y, x) - \tau (y, x) ] / \beta \) so that the operating profit is given by

\[ e^*(y) = \frac{m + k}{\beta} \int_X [ p^*(y, x) - \tau (y, x) ]^2 \lambda(x) dx \]  

(10)

The operating profit reflects a market access and market crowding effect. On the one hand, the firm located at \( y \) benefits from lower transport costs and higher operational profits when consumers locate closer to it. However, the stronger proximity of consumers is associated with a stronger proximity of the other firms and therefore with a more intense competition. Indeed, one can check that the price \( \bar{p}(y) \) falls as more firms locate around the location \( y \).

In the long run, entry occurs until firms earn zero profit. This means that \( \Pi^*(y) = 0 \) in any location \( y \) where a firm can set up its production activity. Using (8), the wage paid by a firm at location \( y \) should then be equal to

\[ w^*(y) = e^*(y) - K + S_k[p^*(\cdot , y)] \]  

(11)

The presence of vertical linkages impacts on the worker’s wage in two ways. First, it increases the operating profit \( e^*(y) \) because her production is sold not only to consumers but also to other firms. Second,
it increases her firm’s profit through the capital savings that are induced by the use of intermediate inputs. Those capital savings take the same form as her own surplus from consumption.

As mentioned above, we impose that product varieties can be exchanged for any pair of locations and for any distribution of firms and consumers. This means that the firms’ prices net of transport costs on a variety produced in $x$ and sold in $y$ are always positive, i.e.,

$$p^*(y, x) - \tau(y, x) > 0 \quad \forall y, x$$

Let $\mathcal{B} \subset \mathcal{X}$ be the support of the city and let the maximal distance between any two points of $\mathcal{B}$ be $2b \equiv \max_{x, y \in \mathcal{B}} T(x - y)$. Then, the possibility of exchanging varieties from any location requires that

$$\tau(2b)^2 < \frac{2\alpha \beta}{2\beta + \gamma}$$

(12)

Note that the feasible exchange condition (12) involves endogenous variable $b$ that must be replaced by the equilibrium value. This is determined in later sections.

### 3.3 Location incentives

In this model, workers reside at their firm’s location. Because of fixed labor requirements, firms and workers have the same spatial distribution. As a result the spatial distribution of firms is driven by the location of workers $\lambda(x)$. The workers’ incentives to reside in some locations are given by their utility differentials. We here collect the above results to establish their utility level when product markets, land market and labor markets clear.

Because each unit of geographical space hosts $\lambda(x)$ firms that each hires a unit mass of individuals, each individual uses $s(x) = 1/\lambda(x)$ units of space. As a result, we can write the consumer-worker’s indirect utility function (5) as

$$V(x) = w^*(x) + mS^*[p^*(\cdot, x)] - \theta \lambda(x) + \tau_0$$

which includes its surplus from consumption, its wage and the residential disutility that more dense locales impose on him through higher land rents. Using the equilibrium wage (11), we finally get

$$V(x) = e^*(x) + (m + k)S^*[p^*(\cdot, x)] - \theta \lambda(x) - K + \bar{c}_0$$

(13)

This expression summarizes the agglomeration and dispersion forces in the model. First, by (10), the firms’ operating profits $e^*$ increase proportionally with demands for both final and intermediate
products (i.e. with $m + k$). Because of free entry, operational profits are shifted to consumers. As we mentioned above, the operational profits are subject to the market access and market crowding effects. When firms and consumers locate closer to each other, firms benefit from lower transport costs but face a tougher competition. Those effects respectively represent an agglomeration and a dispersion force. Second, a higher consumer demand of final goods, $m$, increases consumer surplus $mS^*$ and fosters spatial concentration because consumers’ prices become lower when firms pay lower transport costs on their goods and services. Third, a higher demand for intermediate goods by firms, $k$, increases the firms’ incentives to use more intermediate inputs. Because of free entry, capital cost savings $kS^*$ are shifted to consumers who then have an additional advantage to spatially concentrate. Finally, the consumer’s utility falls with higher residential densities in more spatially concentrated areas, $\theta \lambda$, because of smaller residential lots and higher land rents there.

We finally break down the transport cost from locations $x$ to $y$ as

$$\tau(x, y) \equiv \tau T(x - y)$$

where $\tau$ is the amplitude of transportation costs and $T$ captures the shape of transport costs. Collecting the above results, the consumer-worker’s indirect utility can be rewritten as the following function of $\lambda(\cdot)$ and $x$

$$V(x) = W_0 - W_1 f_1(x) + W_2 f_2(x) - W_3 f_3(x) - W_4 [f_1(x)]^2 - W_5 \lambda(x)$$

where $f_1$, $f_2$ and $f_3$ are three “accessibility measures” defined as

$$f_1(x) \equiv \int_\mathcal{X} T(x - z) \lambda(z) \, dz$$

$$f_2(x) \equiv \int_\mathcal{X} [T(x - z)]^2 \lambda(z) \, dz$$

$$f_3(x) \equiv \int_{\mathcal{X} \times \mathcal{X}} T(x - y) T(y - z) \lambda(y) \lambda(z) \, dy \, dz$$

and where $W_0$ is a constant and

$$W_1 = \frac{\tau \alpha (3\beta + 2\gamma)}{(2\beta + \gamma)^2} (m + k) \quad W_2 = \frac{3\tau^2}{8\beta} (m + k) \quad W_3 = \frac{\tau^2 \gamma}{2\beta(2\beta + \gamma)} (m + k)$$

$$W_4 = \frac{\tau^2 \gamma^2}{8\beta(2\beta + \gamma)^2} (m + k) \quad W_5 = \theta$$

All constants $W_j$’s are all positive and ‘generically’ different from zero in the sense that $W_j > 0$ for any non-zero measure of parameters $(\alpha, \beta, \gamma, \tau, \theta)$ (see Picard and Tabuchi, 2007 and 2010). Note that expression (14) applies for any dimension of the Euclidean space, $\mathcal{X}$. 

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We are now equipped to analyze the long-run spatial equilibrium.

4 Spatial equilibrium

In a spatial equilibrium, workers have no incentives to relocate and therefore get the same utility level everywhere. Formally, a spatial equilibrium is such that $\lambda(x) > 0$ if $V(x) = \bar{V}$ and $\lambda(x) = 0$ if $V(x) < \bar{V}$. To our knowledge, the spatial equilibrium condition $V(x) = \bar{V}$ has no explicit solution for the spatial distribution $\lambda(x)$ for a general class of transport cost functions. Yet, one important class of spatial distribution is readily spotted for the following quadratic transport costs:

$$T(x, y) \equiv \|x - y\|^2 = \begin{cases} (x - y)^2 & \text{if } X = 1 \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 & \text{if } X = 2 \end{cases}$$

where $x$ denotes the coordinates of a consumer, $y$ the coordinates of a firm, and $\|\cdot\|$ the Euclidean distance. Such transport cost functions are commonly used in Hotelling models and its various applications (see Anderson et al. 1992). Economides (1986) has discussed the analytical difficulties and absence of pure strategy equilibrium under non-quadratic transport cost functions. Quadratic transport cost functions imply that travel/shipping costs increase more than proportionally with distance. This is likely to be the case in a city where longer distances imply changes of modes of transportation and higher travel/ship cost. For instance, activities requiring close distance can be done by foot, those requiring longer distance needs to combine foot and metro or buses whereas much longer distance requires additional change of metro or buses.\(^5\)

Under quadratic costs, the accessibility measures become

$$f_1(x) = \int_B \|x - z\|^2 \lambda(z) \, dz$$
$$f_2(x) = \int_B \|x - z\|^4 \lambda(z) \, dz$$
$$f_3(x) = \int_{B \times B} \|x - y\|^2 \|y - z\|^2 \lambda(y) \lambda(z) \, dy \, dz$$

which are polynomials of $x$ of order 2, 4 and 2, respectively. Similarly, the expression $[f_1(x)]^2$ is a polynomial of $x$ of order 4. As a result, because the spatial equilibrium imposes $V(x) = \bar{V}$ $\forall x \in B$, the spatial equilibrium distribution of workers $\lambda(x)$ must also be a polynomial of order 4.

\(^5\)Analytical results cannot be obtained in the case of linear transport costs. Nevertheless, we have confirmed that the similar results can be obtained by numerical analysis assuming a discrete distribution of workers.
Lemma 1 Under quadratic transport costs, the equilibrium distribution of workers $\lambda(x)$ is a polynomial of order 4.

In the case of a unidimensional Euclidean space $(X = 1)$, the equilibrium distribution of workers is equal to

$$\lambda(x) = \sum_{k=0}^{4} a_k x^k$$

where $a_k \in \mathbb{R}$. In the case of a bidimensional Euclidean space $(X = 2)$, it is equal to

$$\lambda(x_1, x_2) = \sum_{k=0}^{4} \sum_{\ell=0}^{k} a_{k\ell} x_1^\ell x_2^{k-\ell}$$

where $a_{k\ell} \in \mathbb{R}$.

As a corollary, the support $B$ of the city must be a bounded and convex set. Indeed, the fact that $\lambda(x)$ is a polynomial implies that $\{x : \lambda(x) > 0\}$ is a convex open set and the same fact combined with $\int_B \lambda(x) dx = 1$ implies that $\{x : \lambda(x) > 0\}$ is a bounded set.

5 Linear city

We now study the case of a linear city that spreads on the space $X = \mathbb{R}$. We normalize the city width to unity so that $\lambda(x)$ measures the density of workers residing at location $x \in \mathbb{R}$. In other words, any rectangular space $1 \times dx$ of the city includes $\lambda(x) dx$ workers. Let the support of the city be $B = (-b, b)$ where $\pm b$ are the city borders, so that workers are distributed about the location $x = 0$. We assume that the (farming) opportunity cost of land is nil. So, the land rent is nil at the city border, which implies that $\lambda(b)$ is equal to 0.

The spatial equilibrium is defined by a spatial distribution function $\lambda^*(x)$, a city border $b^*$ and a utility level $V^*$ that solve the three equalities $V(x) = V^*, \lambda^*(b^*) = 0$ and $\int_{-b}^{b} \lambda^*(x) dx = 1$. The first equality implies that

$$V'(x) = 0, \quad V''(x) = 0, \quad V'''(x) = 0, \quad V''''(x) = 0 \quad \forall x \in B$$

which can be applied at $x = 0$ to infer the coefficients $(a_0, a_1, a_2, a_3, a_4)$ that define the polynomial $\lambda(x)$ in (16). The equilibrium is therefore obtained in the following way. First, it can be readily shown that the equalities $V'(x) = 0$ and $V''(x) = 0$ imply that the coefficients $a_1$ and $a_3$ are both equal to
zero, which confirms that the spatial distribution is symmetric about $x = 0$. Second, simultaneously solving $V''(x) = 0$, $\lambda(b) = 0$ and $\int_{-b}^{b} \lambda(x) dx = 1$ we can get the following quartic spatial distribution of workers:

$$\lambda^*(x) = \phi \left[ (x^2 - \rho_1)^2 - (\rho_2)^2 \right]$$

(18)

where

$$\rho_1 \equiv -\frac{a_2}{2a_4} = \frac{15 + 24b^5}{40b^2 \phi}$$

and

$$\rho_2 \equiv \frac{\sqrt{a_2^2 - 4a_4 \phi}}{2 \phi} = \frac{15 - 16b^5 \phi}{40b^2 \phi} < \rho_1$$

and

$$\phi \equiv \frac{6\beta^2 + 6\beta \gamma + \gamma^2 \tau^2}{4 \beta (2\beta + \gamma)^2} \frac{\theta(m + k)}{}$$

(19)

Finally, plugging (18) into $V''(x) = 0$, we get

$$g(b) \equiv 1575 \beta^2 (2\beta + \gamma)^4 \theta^2 - 2100 \alpha \beta^2 (2\beta + \gamma)^2 (3\beta + 2\gamma) \theta \tau (m + k) b^3$$

$$+ 210 \beta (2\beta + \gamma)^2 (36\beta^2 + 32\beta \gamma + 5\gamma^2) \theta \tau^2 (m + k) b^5$$

$$- 32 (6\beta^2 + 6\beta \gamma + \gamma^2) (9\beta^2 + 7\beta \gamma + \gamma^2) \tau^4 (m + k)^2 b^{10}$$

(20)

Then, the equilibrium city border $b^*$ is determined by the solution of $g(b^*) = 0$. Expression (20) is 10th-order polynomial that is positive for $b = 0$ and negative for $b \to \infty$. As a result, it accepts at least one positive root. We show in Appendix 1 that this positive root is the unique spatial equilibrium.

We need to check the condition under which exchanges between any city locations are feasible at the equilibrium. Using $g(b) = 0$ and (12), we can explicitly solve them $\tau$ for as follows:

$$\tau < \frac{\alpha^5 (m + k)^2}{\theta^2} G(\beta, \gamma)$$

(21)

where $G(\beta, \gamma)$ is a positive function given in Appendix 1. Because $G(\beta, \gamma)$ is shown to be decreasing in $\gamma$, exchanges are likely to be feasible between any locations if goods are sufficiently differentiated, transport costs are small, or consumers have intense preferences towards the varieties or weak preferences for residential space. We assume condition (21) in the sequel of this section.

We can now discuss the urban structure properties. On the one hand, note that, because $\lambda^*(b^*) = 0$, the spatial distribution $\lambda^*(x)$ cannot be increasing at the city border $x = b^*$. Then, because $\lambda''(x) = 4\phi x (x^2 - \rho_1)$,

$$\lambda''(b^*) \leq 0 \iff b^{*2} \leq \rho_1 \iff b^{*5} \leq \frac{15}{16\phi}$$

(22)

so that the city border $b^*$ cannot exceed $\sqrt[5]{15/16\phi}$. This naturally implies that the spatial distribution of workers and firms falls ($\lambda''(x) \leq 0$) for any $x \in [0, b^*]$. The spatial distribution is then necessarily
single-peaked. Furthermore, because \( \lambda''(x) = 4 \phi (3x^2 - \rho_1) \), the land rent is concave for any \( x \in [0, \sqrt{\rho_1/3}] \). In Appendix 2 we show that \( b^* < \sqrt{\rho_1/3} \) for any spatial equilibrium satisfying the feasible exchange condition (21). As a result the spatial distribution is always concave. Based upon the foregoing, we establish the following.

**Proposition 1** The equilibrium distribution \( \lambda^*(x) \) in a linear city is unique, concave and symmetric about \( x = 0 \).

Intuitively, this means that there always exists a monocentric city (with one bump centered on \( x = 0 \)) contained in the interval \([-b^*, b^*]\). This also means that there exists no polycentric cities (with more than one bump) if varieties of products or services are accessible from everywhere in the city. In this model, polycentric cities could therefore occur under violation of the feasible exchange condition (12) such that some products and services are not accessible in some city locations. See Figure 1 for an equilibrium workers’ distribution given the parameter values of \( \alpha = \beta = \gamma = 1 \) and \( \theta = \tau = 1/10 \).

Insert Figure 1 here

Next, the equilibrium land rent (7) is rewritten as

\[
R^*(x) = \frac{\theta}{2} \lambda^*(x)^2
\]

Because the workers’ distribution given by (18) is single-peaked and symmetric about \( x = 0 \) from Proposition 1, the land rent is also single-peaked and symmetric about the city center. Furthermore, we have \( R^*(\pm b) = R''(\pm b) = 0 \) and

\[
R''(0) = -\frac{3\theta}{200b^4} (4\phi b^5 + 15) (8\phi b^5 + 5) < 0
\]
\[
R''(\pm b) = \frac{\theta}{100b^4} (16\phi b^5 - 15)^2 > 0
\]

implying that the land rent is concave near the city center and convex near the city edges. That is, the equilibrium land rent is bell-shaped, whereas the equilibrium workers’ distribution is concave. The concave part of the rent function corresponds to the completely integrated configuration obtained in Ogawa and Fujita (1980, Figure 3) although the latter allow firms and workers to choose different locations. The concavity near the city center results from the fact that the access to all other firms and
consumers are convex in $x$. The convex part of the rent function is consistent with that in standard textbooks of urban economics à la Alonso (1964) and is explained by the substitution between access and space for land.

We now turn to the comparative statics of the equilibrium urban structure.

**Comparative statics** The residential density at the city center is given by $\lambda(0) = a_0 = (4\phi b^5 + 15) / (20b)$. Differentiating it with respect to $b$, we get

$$\frac{\partial \lambda(0)}{\partial b} = \frac{16\phi b^5 - 15}{20b^2} < 0$$

where the inequality is due to (22). Hence, the residential density at the city center is inversely related to the city spread $2b$.

Using the implicit function theorem and employing the result $\partial g(b^*) / \partial b < 0$ obtained in Appendix 1, we can derive the following comparative statics for the city border $b$:

$$\frac{db^*}{d\tau} < 0, \quad \frac{db^*}{d\alpha} < 0, \quad \frac{db^*}{d\gamma} > 0 \quad \text{and} \quad \frac{db^*}{d\theta} > 0 \quad (23)$$

See Appendix 3 for the proof. First, a fall in the transport cost $\tau$ increases the city spread and makes the city center less dense. In this case, goods or services can be offered from further locations so that residents can benefit from both a larger residential space and higher earnings because of the weaker competition. This result is congruent to Helpman’s (1998) model, where space is used for housing consumption. Second, more intense preferences towards the varieties $\alpha$ reduce the city spread and make the city center more dense. As residents consume more of each variety, firms and workers can increase their production and hence profits and wages by locating closer to each other. Third, stronger product substitution between varieties $\gamma$ implies a larger city spread and a lower density near the city center. On the one hand, stronger product substitution entices firms to use space as a way to differentiate themselves and reduce competition. On the other hand, it allows residents to swap their consumption of distant goods or services for residential space. Finally, stronger preferences for residential space $\theta$ increase the city spread and decrease the residential density at the center. This corresponds to the so-called suburbanization process experienced in many large cities. The comparative statics with respect to the four parameters agree with the findings in new economic geography and here apply to urban structures:
Proposition 2 The population density near the city center falls and the city borders expand for lower transport cost ($\tau$ smaller), less intense preference towards varieties ($\alpha$ smaller), stronger product substitutability ($\gamma$ larger), and stronger preference for residential space ($\theta$ larger).

The effects of demand linkages $m$ and input-output linkages $k$ are opposite to the preference for residential space $\theta$. This is because the weights ($W_1, ..., W_4$) are proportional to $m + k$ and the weight $W_5$ is proportional to $\theta$ in (15). While preferences for residential space generate a dispersion force, stronger demand and input-output linkages (through higher $m + k$) generate an agglomeration force. We summarize this result in the following proposition.

Proposition 3 Stronger demands for final products or services and/or for intermediate products or services leads to stronger final demand linkages and/or stronger vertical linkages, which both increase the density of firms near the city center and reduce the city spread.

This analysis reflects the individual’s trade-off between her demand for residential space, her consumption and her income. Both effects on consumption and production tend to increase firms’ and workers’ concentration in the vicinity of the city center. When $m$ increases, individuals have larger demands for products or services. This entices firms to save transport costs by locating closer to consumers and to pull their workers closer to the city center. When $k$ increases, firms are able to make larger savings in capital requirements and therefore have larger demand for intermediate products or services. Larger demands for intermediate inputs entice firms to locate closer to each other and save on transport costs. So, firms can raise their operating profits through larger capital savings and through a better proximity to their clients. Finally, because labor markets clear, firms’ profits are shifted to workers whose higher incomes are used to pay the higher residential rents near the city center. Note that, under the present specification, final demand linkages and vertical linkages have the same effects on firms’ locations.

So far, we have been assuming that the geographical space is unlimited. One may wonder about the equilibrium city structure when the space is limited to the line segment $[-\overline{b}, \overline{b}]$ where $\overline{b}$ is exogenous. Obviously, the equilibrium city structure does not change when the exogenous border $\overline{b}$ is larger than the above equilibrium border $b^*$. By contrast, if $\overline{b}$ is smaller than $b^*$, the city has not enough space to expand so that the opportunity cost of land can no longer be zero and the population density

\[\text{We thanks a referee for this interesting extension.}\]
can no longer fall to zero at the city border. The equilibrium conditions become $V(x) = V^*$ and $\int_{-b}^{b} \lambda(x) dx = 1$. In fact, the equilibrium condition $\lambda^*(b^*) = 0$ disappears at the same time as the endogenous variable $b^*$. It can be shown that the equilibrium solution $\lambda^*(x)$ is still given by the quartic expression (18) with exogenous $b = \bar{b}$ and with two positive constants $\rho_1$ and $\rho_2$. As a result, the equilibrium distribution $\lambda^*(x)$ is positive, symmetric and single-peaked on the line segment $[-\bar{b}, \bar{b}]$.

One may also think about extending this analysis to non-Euclidean space such as the perimeter of a circumference. Note that, contrary to the open or closed line segment, this spatial structure is fully symmetric. As a consequence, there will always exist a symmetric equilibrium where firms and workers evenly spread around the circumference. This is what the literature denotes as a “flat earth” equilibrium. The main question is whether such an equilibrium is robust to some stability criterion. Using Picard and Tabuchi’s (2010) stability analysis, one can show that this equilibrium will be robust to small perturbations of the equilibrium distribution if preferences for space are large enough (large enough $\theta$) and/or if the perimeter of the circumference is small enough. This is intuitive because agents with high demands for residential plots are enticed to cover the full space available. For small preferences for space or large perimeters, the symmetric equilibrium will not be stable. In this case, for instance, if the perimeter of a circle is longer than $4\bar{b}^*$, the urban structure characterized in Proposition 1 will also be a spatial equilibrium. For intermediate length perimeters, there could be multiple equilibria with a set of equidistant and identical cities like the ones discussed in Mossay and Picard (2011).

6 Planar city

We now study the case of a city that spreads on the bidimensional Euclidean space $\mathcal{X} = \mathbb{R}^2$. Without loss of generality, we suppose that workers are distributed about the location $x = 0$ and we let $\lambda(x)$ measure the density of workers residing at location $x \in \mathbb{R}^2$. We here show that the circular city is a spatial equilibrium. Towards this aim we replace the Cartesian coordinates $(x_1, x_2)$ by the polar coordinates $(r, \varphi)$ where $r$ is the distance of point $x$ to the origin $(0, 0)$ and where $\varphi$ is the respective polar angle with the horizontal axis $Ox_1$. An infinitesimal unit of space $dx_1 dx_2$ must be converted to $r d\varphi dr$ under polar coordinates. Following Lucas and Rossi-Hansberg (2001), we assume a circular city such that individuals’ density function is expressed as $\lambda(r, \varphi) \equiv \lambda(r)$. The support of a circular city is
\( B = [0, b] \times [-\pi, \pi] \) where \( 2b \) is equal to the city diameter.

The first accessibility measure can successively be written as

\[
f_1(r, \phi) = \int_B \left[ (s \cos \phi - r \cos \phi)^2 + (s \sin \phi - r \sin \phi)^2 \right] \lambda(s) \ s \ d\phi \ ds
\]

\[
= \int_B \left[ s + r - 2sr \cos (\phi - \phi) \right] \lambda(s) \ s \ d\phi \ ds
\]

\[
= 2\pi \int_0^b (r^2 + s^2) \lambda(s) \ s \ ds
\]

where \((s, \phi) \in B\) are polar coordinates for the integration variables. By the same argument, the second and third accessibility measures can be computed as

\[
f_2(r, \phi) = 2\pi \int_0^b (r^4 + 4r^2s^2 + s^4) \lambda(s) \ s \ ds
\]

\[
f_3(r, \phi) = 4\pi^2 \int_0^b \int_0^b \left[ \left( r^2 + s^2 \right) \left( r^2 + t^2 \right) \right] \lambda(s)\lambda(t) \ s \ ds \ dt
\]

\( s \) and \( t \in [0, b] \) are distance-to-origin coordinates of the integration variables. Obviously, those accessibility measures depend only on \( r \). Because the spatial distribution \( \lambda(r) \) of the circular city also depends only on \( r \), the indirect utility \( V(r, \phi) \) depends only on \( r \). The circular city is therefore consistent with a spatial equilibrium.

The urban structure of the circular city is derived in the same way as in the linear city. Note that, by Lemma 1, spatial distributions are 4th-order polynomials of the Cartesian coordinates \((x_1, x_2)\). In addition, to satisfy circular symmetry, this should be a polynomial function of \( r^2 = x_1^2 + x_2^2 \); that is, it should have the following form: \( a_4(x_1^2 + x_2^2)^2 + a_2(x_1^2 + x_2^2) + a_0 \). So, the class of spatial distributions that satisfy that is 4th-order polynomials and has circular symmetry is given by \( \lambda(r) = a_4r^4 + a_2r^2 + a_0 \). As a result, the equilibrium spatial distributions \( \lambda(r) \) must also have a zero slope at its peak \( (\lambda'(0) = 0) \). This property results from the combination of circular symmetry and quadratic transport costs (which drives Lemma 1).

We assume again that the (farming) opportunity cost of land is nil so that the land rent is nil at the city border: \( \lambda(b) = 0 \).

The spatial equilibrium is then defined by a spatial distribution function \( \lambda^*(r) \), a city border \( b^* \) and a utility level \( V^* \) that solve the three equalities \( V(r) = V^*, \lambda^*(b^*) = 0 \) and \( \int_{-\pi}^{\pi} \int_{-b}^{b} \lambda^*(r)r d\phi dr = 1 \) where the last integral is equal to \( 2\pi \int_{-b}^{b} \lambda^*(r)r dr \). As before, the first equality can be used to give the following necessary conditions

\[
V'(r) = 0, \ V''(r) = 0, \ V'''(r) = 0, \ V''''(r) = 0 \ \forall r \in [0, b]
\]
which can be applied at \( r = 0 \) to find the coefficients \((a_0, a_2, a_4)\). The equilibrium is obtained in the same way as in the case of a linear city, except that the conditions \( V'(r) = 0 \) and \( V'''(r) = 0 \) are already satisfied because of the circular symmetry. On the one hand, simultaneously solving \( V'''(x) = 0, \lambda(b^*) = 0 \) and \( 2\pi \int_{-b}^{b} \lambda^*(r)r\,dr = 1 \), we get the workers’ spatial distribution \( \lambda^* \) given by expression (18) where the parameters are now given by

\[
a_0 = \frac{\pi \phi b^6 + 6}{3\pi b^2}, \quad a_2 = -\frac{4\pi \phi b^6 + 6}{3\pi b^4} \quad \text{and} \quad a_4 = \phi
\]

and where \( \phi \) is defined exactly as in (19). On the other hand, plugging (24) into \( V''(x) = 0 \), the city border is given by the equality \( g(b) = 0 \) where

\[
g(b) = 1152\beta^2(2\beta + \gamma)^4\theta^2 - 576\pi\alpha \beta^2(2\beta + \gamma)^2(3\beta + 2\gamma)\theta \tau (m + k)b^4
\]

\[+ 48\pi \beta (2\beta + \gamma)^2 \left[ 48\beta^2 + 8(6 - \tau) \beta \gamma + (9 - 4\tau) \gamma^2 \right] \theta \tau^2 (m + k)b^6
\]

\[- \pi^2 \left( 6\beta^2 + 6\gamma \beta + \gamma^2 \right) \left[ 24\beta^2 + 16\beta \gamma + \gamma^2 \right] \tau^4 (m + k)^2 b^{12}\]

which is a polynomial equation of order 12. This solution is very similar to the one obtained for the linear city. The existence and uniqueness of the equilibrium border \( b \) are given by the same argument as above and relegated to Appendix 1. So, we can make a proposition similar to Proposition 1.

**Proposition 4** There exists a concave two-dimensional equilibrium distribution \( \lambda^* (x) \) with circular symmetry, single peak and zero slope at its center.

Although we have shown that \( \lambda^* (x) \) is concave and circular symmetric, there might exist other equilibrium distributions under circular asymmetry such that \( \lambda(r, \varphi) \neq \lambda(r) \).

See Figure 2 for an equilibrium workers’ distribution given the parameter values of \( \alpha = \beta = \gamma = m = k = 1 \) and \( \theta = \tau = 1/10 \). It can be verified that the comparative static properties are the same as in the case of a linear city. Because the urban structures of linear and planar cities have very similar properties, the present analysis can be seen as a validation for the generality of studies that focuses on linear cities.

**Insert Figure 2 here**
7 Conclusion

To our knowledge this paper presents the first formal discussion of the endogenous urban structure of a city that is subject to endogenous backward and forward linkages. Workers consume the goods or services they produce while they must rent their residential lots in an urban land market. Firms produce under increasing returns to scale and sell their differentiated goods in markets where demand and supply balance. As in most modern cities, firms are subject to vertical linkages, which we model in the spirit of Krugman and Venables (1995). The production and market structures are exactly the same as those found in standard new economic geography models (Krugman, 1991; Ottaviano et al., 2002).

We showed that firms and workers co-agglomerate about a city center symmetrically with a single peak. We examined the shape of the residential distribution in such cities on a one- and two-dimension Euclidean space (linear city and planar city). We verified that the effects of final demand and vertical linkages are complement and opposite to the preference for residential space.

The present model therefore confirms a series of properties that are known in the existing urban literature or expected from the new economic geography literature. The present paper also shows that the combination of specific transport costs and preferences for space offers a good level of analytical tractability. This parallels the tractability properties of address models of spatial competition (Anderson et al. 1992), where the transport cost is a quadratic function of distance, but where consumers are nevertheless immobile. As those properties permit to determine spatial equilibrium conditions in one- or two-dimensional Euclidean spaces without the recourse to numerical exercises, they also offer some research perspectives for extensions in various directions such as socially optimal allocations of firms, labor market specificities, system of cities and possibly endogenous commuting. In particular, multiple city centers are likely to arise when firms incur a too high shipping cost to distant places or when workers are able to commute to firms. At the present stage, those extensions are left for further research.
Appendices

Appendix 1

We here prove the uniqueness of the city border $b$ in the cases of linear and planar cities.

**Linear city** First, note that at the city border $b$, it must be that $\lambda'(b) \leq 0$. Indeed, if $\lambda'(b) > 0$, this implies that $\lambda(b - \varepsilon) < 0$ for any sufficiently small positive $\varepsilon \in [0, b]$, a contradiction. Hence, by (22), the city border $b$ must be lower than $\sqrt[3]{15}/16\phi$.

Second, since $g(0) > 0$ and $g(\infty) < 0$, there exists at least one positive root $b^*$ of $g(b) = 0$. The smallest root must naturally have $g'(b^*) < 0$.

Third, differentiating $g(b)$ with respect to $b$ and substituting the solution in $\alpha$ of $g(b) = 0$ (which is linear in $\alpha$) the result yields

$$g'(b^*) = -\frac{C_1}{b^*} \left( b^{*5} - \frac{15}{16\phi} \right) \left[ b^{*5} - \frac{15}{16\phi} (1 + C_2) \right]$$

where $C_1$ and $C_2$ are positive constants. This derivative alternates its sign for different roots $b^*$. The smallest positive root lies below $\sqrt[3]{15}/16\phi$, the second positive root lies between $\sqrt[3]{15}/16\phi$ and $\sqrt[3]{15}(1 + C_2)/16\phi$ and the last root above $\sqrt[3]{15}(1 + C_2)/16\phi$. Therefore, there exists only one root $b^*$ with $b^* < \sqrt[3]{15}/16\phi$.

Finally, this root is acceptable only if exchanges between any city locations are feasible at the equilibrium. Let $\hat{b} \equiv \sqrt[3]{\frac{2\alpha^3 \beta^\gamma + 1}{2\beta + \gamma}}$ be the upper limit of the feasible border. Because $g'(b) < 0$, it must be that $b^* < \hat{b} \iff g(b^*) > g(\hat{b})$. Using $g(b^*) = 0$, we get

$$g(b^*) = 0 > g(\hat{b}) = C_3 \left( \sqrt{\tau} + C_4 \right) \left( \sqrt{\tau} - C_5 \right)$$

where $C_3$, $C_4$, and $C_5$ are positive constants. This can be simplified as

$$\sqrt{\tau} < C_5^2$$

which is equivalent to (21) where

$$G(\beta, \gamma) \equiv \frac{7\beta^3 \left[ \sqrt{7}(84\beta^2 + 108\beta\gamma + 35\gamma^2) + \sqrt{C_6} \right]^2}{352800 \left( 2\beta + \gamma \right)^2}$$

and $C_6 \equiv 51120\beta^4 + 130080\beta^3\gamma + 124632\beta^2\gamma^2 + 53336\beta\gamma^3 + 8607\gamma^4$. It can be shown that $G(\beta, \gamma)$ is positive and decreasing in $\gamma$. 

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Planar city  We use the same argument to that the equilibrium is unique in the case of planar city. We know that \( \lambda'(b) \leq 0 \). The counterpart of (22) is given by
\[
\lambda'(b) < 0 \iff b < \frac{\sqrt{3}}{\pi \phi}
\] (26)
Plugging the solution in \( \alpha \) of \( g(b) = 0 \) into \( g'(b) \) yields
\[
g'(b^*) = -C_7 \left( b^* - \frac{3}{\pi \phi} \right) \left[ b^* - \frac{3}{\pi \phi} \left( 1 + C_8 \right) \right]
\]
where \( C_7 \) and \( C_8 \) are positive constants. Because the derivative alternates its sign for different \( b^* \) and because the derivative is negative for all \( b^* \in (0, \sqrt{3/\pi \phi}) \), there exists a unique \( b^* \).

Appendix 2
In this appendix we prove that the equilibrium spatial density \( \lambda(x) \) is a concave function under the feasible exchange condition (12) in the cases of linear and planar cities.

Linear city  From (18), we immediately get \( \lambda'''(x) < 0 \). Because \( \lambda'''(x) \) is decreasing and \( \lambda'''(0) \) is positive, it is sufficient to exclude the possibility of \( \lambda'''(b^*) \geq 0 \). Under this condition, \( \lambda'''(x) < 0 \) holds for all \( x \in [0, b^*] \).

It is convenient to introduce
\[
\xi^* = \frac{2 \beta + \gamma}{\alpha \beta} b^* \tau
\] (27)
where \( \xi^* < 1 \) from the feasible exchange condition (12). Suppose there exists \( b^* \) that satisfies \( \lambda'''(b^*) = 0 \) and \( \xi^* < 1 \). Then, solving (27) and \( \lambda'''(b^*) = 0 \) simultaneously, we can express \( (b^*, \tau^*) \) as a function of \( \xi^* \):
\[
b^* = \frac{5 (2 \beta + \gamma)^2 \theta}{2 \alpha \beta (6 \beta^2 + 6 \beta \gamma + \gamma^2) (m + k) \xi^2}
\]
and
\[
\tau^* = \frac{2 \alpha \beta^3 (6 \beta^2 + 6 \beta \gamma + \gamma^2)^2 (m + k)^2 \xi^5}{25 (2 \beta + \gamma)^9 \theta}
\] (28)
Plugging (28) into (20), we get
\[
g(b^*) = \frac{125 \beta^2 (2 \beta + \gamma)^4 \theta^2}{4 (6 \beta^2 + 6 \beta \gamma + \gamma^2) \xi^*} \left[ 18 (25 \xi^* - 28) \beta^2 + 14 (31 \xi^* - 42) \beta \gamma + 5 (71 \xi^* - 168) \gamma^2 \right] < 0
\] (29)
for all \( \xi^* < 1 \), which violates the equilibrium condition of \( g(b^*) = 0 \). This implies that there does not exist \( b^* \) simultaneously satisfying \( \lambda'''(b^*) = 0 \) and \( \xi^* < 1 \). That is, \( \lambda'''(b^*) < 0 \) holds, and thus, \( \lambda'''(x) < 0 \) holds for all \( x \in [0, b^*] \). Because \( \lambda^*(x) \) is symmetric about a single peak at \( x = 0 \) from (18), it is concave.
Planar city  We use the same argument as that of linear city. The corresponding expressions of (28) and (29) are

\[ b^* = \frac{4 (2\beta + \gamma)^2}{2\alpha\xi^*} \sqrt{\frac{3\theta}{7\pi\beta (6\beta^2 + 6\beta\gamma + \gamma^2) (m+k)}} \quad \text{and} \quad \tau^* = \frac{7\pi\alpha^3 \beta^2 (6\beta^2 + 6\beta\gamma + \gamma^2) (m+k)\xi^*3}{96 (2\beta + \gamma)^5 \theta} \]

\[ g(b^*) = \frac{432\beta^2 (2\beta + \gamma)^4 \theta^2}{49 (6\beta^2 + 6\beta\gamma + \gamma^2) \xi^*} \left[ 24 (51\xi^* - 56) \beta^2 + 32(36\xi^* - 49)\beta\gamma + (177\xi^* - 448)\gamma^2 \right] < 0 \]

Hence, \( \lambda^*(x) \) is concave.

Appendix 3

In this appendix we prove the comparative static results (23). Applying the implicit function theorem to \( g(b^*) = 0 \) for any parameter \( \xi \in \{\alpha, \theta, \tau, \gamma\} \), we have

\[ \frac{db^*}{d\xi} = -\frac{\partial g(b^*)/\partial \xi}{\partial g(b^*)/\partial b} \]

where \( \partial g(b^*)/\partial b < 0 \) from Appendix 1. Therefore \( db^*/d\xi < 0 \) if and only if \( \partial g(b^*)/\partial \xi < 0 \). First, we note that \( db^*/d\alpha < 0 \) because expression (20) trivially falls with \( \alpha \) so that \( \partial g(b^*)/\partial \alpha < 0 \). Second, we readily get that \( db^*/d\theta > 0 \). Indeed, solving \( g(b) = 0 \) for parameter \( \alpha \) and plugging it into \( \partial g(b^*)/\partial \theta \), we have

\[ \frac{\partial g(b^*)}{\partial \theta} \bigg|_{g(b)=0} = 1575\beta^2 (2\beta + \gamma)^4 \theta + 32 (6\beta^2 + 6\beta\gamma + \gamma^2) (9\beta^2 + 7\beta\gamma + \gamma^2) \tau^4(m+k)2b^{10}/\theta \]

which is positive. Hence, we have proved that \( db^*/d\theta > 0 \). Third, we show that \( db^*/d\gamma > 0 \). Solving \( g(b) = 0 \) for parameter \( \alpha \) and plugging it into \( \partial g(b^*)/\partial \gamma \), we have

\[ \frac{\partial g(b^*)}{\partial \gamma} \bigg|_{g(b)=0} = h \left[ (b^*)^5 \right] \quad \text{where} \quad h(z) = \frac{3150\beta^2 (2\beta + \gamma)^4 \theta^2 + C_9 z - C_{10} z^2}{(2\beta + \gamma) (3\beta + 2\gamma)} \]

where \( C_9 \) and \( C_{10} \) are positive constants. Because \( h(z) \) is concave in \( z \), \( h(0) > 0 \), and \( h(15/16\phi) > 0 \), we get \( h(z) > 0 \) for all \( z = (b^*)^5 \in (0, 15/16\phi) \). Hence, \( db^*/d\gamma > 0 \). Finally, we show that \( db^*/d\tau < 0 \). We indeed get

\[ \frac{\partial g(b^*)}{\partial \tau} = 4b^3 (m+k) \left[ -525\alpha\beta^2 (2\beta + \gamma)^2 (3\beta + 2\gamma) \theta + C_{11} \tau b^2 - C_{12} \tau^3 b^{*7} \right] \]

where \( C_{11} \) and \( C_{12} \) are positive constants. It can be checked that \( \partial g(b^*)/\partial \tau \) is negative at the maximum feasible exchange transport cost \( \tau = 2\alpha\beta/ [(2\beta + \gamma) (2b^*)^2] \). Furthermore, \( \partial g(b^*)/\partial \tau \) is increasing in
\( \tau \) for all feasible \( \tau \) because

\[
\frac{\partial^2 g(b^*)}{\partial \tau^2} = 4b^* (m + k) \left( C_{11} - 3C_{12} \tau^2 b^* \right) > 0
\]

is positive. This follows from the condition \( b^* < \frac{15}{16\phi} \) as already readily shown and from the fact that one can further show that \( 15/(16\phi) < C_{11}/(3C_{12} \tau^2) \). As a result, \( \partial g(b^*)/\partial \tau \) is increasing in \( \tau \) for all feasible \( \tau \). Thus, \( db^*/d\tau < 0 \) holds.

References


Figure 1: Equilibrium workers' distribution in a linear city

Figure 2: Equilibrium workers' distribution in a planar city